

15/10/2015

الخميس

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محاضرة [3]

Properties of Convolution:

① Commutative Property of Convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x(n-k) h(k)$$

Proof:-

assume $n-k=m$

$$k = n-m$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$= \sum_{m=-\infty}^{\infty} x(n-m) h(m)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k) h(k)$$

$$k = -\infty \Rightarrow m = \infty$$

$$k = \infty \Rightarrow m = -\infty$$

هذه الخاصية تفيد في إثبات
الخاصية الثانية

② Causality of LTI:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)$$

$$= \sum_{k=-\infty}^{-1} h(k) x(n-k) + \sum_{k=0}^{\infty} h(k) x(n-k)$$

$$= \left[\cancel{h(-1)} x(n+1) + \cancel{h(-2)} x(n+2) + \cancel{h(-3)} x(n+3) + \dots \right]$$

$$+ \left[h(0) x(n) + h(1) x(n-1) + h(2) x(n-2) + \dots \right]$$

System is
causal when
 $h(n)=0, n < 0$

The system is causal if:

$$h(n) = 0, n < 0$$

①

Ex: ① $h(n) = \{1, -2, \underset{\uparrow}{0}, 1/2, 3\} \Rightarrow \text{non-causal}$
($h(n) \neq 0, n < 0$)

② $h(n) = \{1, -2, 0, 1/2, 3\} \Rightarrow \text{Causal}$

③ Stability of LTI:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)$$

$$\text{if } \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Then the System is stable

Z-Transform:

- Cont. time system:-

time domain $\xrightleftharpoons[\text{LT}^{-1}]{\text{LT}}$ s-domain

- discrete time system

discrete time domain $\xrightleftharpoons[\text{Z}^{-1}]{\text{Z.T.}}$ Z-domain

$$z = e^{-Ts}, \quad T = \text{Sampling Time}$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n-k)$$

if $T \neq 1$

$$x(nT) = \sum_{k=-\infty}^{\infty} x(kT) \delta(n-k)T$$

\Downarrow L.T.

$$\begin{aligned} \mathcal{L}[x'(t)] &= \mathcal{L}[x(nT)] \\ &= \sum_{k=-\infty}^{\infty} x(kT) e^{-kTs} \end{aligned}$$

$$\begin{array}{c} x(t) \searrow \\ \quad \quad \quad \downarrow T \\ \quad \quad \quad x^*(t) \equiv x(nT) \end{array}$$

$$X^*(s) = \sum_{k=-\infty}^{\infty} x(kT) \underbrace{e^{-kTs}}_{\Rightarrow (e^{-Ts})^k}$$

$z, T \downarrow$

$$z = e^{-Ts}$$

$$X(z) = \sum_{k=-\infty}^{\infty} x(kT) z^{-k}$$

$$x(n) \xrightarrow{z, T} X(z)$$

for a discrete time sequence:

$$x(nT) \equiv x(n) \quad | \quad T=1 \text{ sec}$$

the z . transform for this sequence is

$$Z[x(n)] = X(z) = \sum_{k=-\infty}^{\infty} x(kT) z^{-k}$$

$$= X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad (T=1 \text{ sec})$$

Ex: Find z .T for $\delta(n)$

Solution

$$Z[\delta(n)] = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

for causal signals and system

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

we deal with LTI systems and causal system.

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \underbrace{x(0)}_1 + \underbrace{x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \dots}_0$$

$$x(n) = \delta(n) = \begin{cases} 1, & n=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\mathcal{Z}[\delta(n)] = 1$$

Ex: find Z.T for $x(n) = u(n)$

Solution: -

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + \dots$$

متوالية هندسية منتهية
الحد الثابتة (z^{-1})

$$= \frac{1}{1 - z^{-1}}$$

$$z^{-1} < 1 \Rightarrow |z| > 1$$

$$X(z) = \frac{z}{z-1} \text{ for } x(n) = u(n)$$

* Region of convergence (ROC):

$$\text{ROC: } z^{-1} < 1$$

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$$\Rightarrow |z| > 1$$

Ex2: find Z.T for $x(n) = e^{-an}$

Solution:

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-an} z^{-n} = 1 + e^{-a} z^{-1} + e^{-2a} z^{-2} + \dots$$

$$e^{-a} z^{-1} < 1$$

$$= \frac{1}{1 - e^{-a} z^{-1}}$$

$$X(z) = \frac{z}{z - e^{-a}}$$

$$\text{Roc: } e^{-a} z^{-1} < 1 \quad \text{or} \quad e^a z^1 > 1$$

Ex 3: $x(n) = n$, find Z.T.:-

$$\text{Solution: } X(z) = \sum_{n=0}^{\infty} n z^{-n}$$

$$X(z) = 0, z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \dots \quad \text{--- (1)}$$

$$z^{-1} X(z) = z^{-2} + 2z^{-3} + 3z^{-4} + 4z^{-5} + \dots \quad \text{--- (2)}$$

$$X(z) - z^{-1} X(z) = z^{-1} + z^{-2} + z^{-3} + z^{-4} + \dots$$

$$[1 - z^{-1}] X(z) = \frac{z^{-1}}{1 - z^{-1}}$$

$$\left| \begin{array}{l} \text{Roc: } z^{-1} < 1 \\ \text{or } z > 1 \end{array} \right.$$

$$X(z) = \frac{z^{-1}}{(1 - z^{-1})^2} = \frac{z}{(z - 1)^2}$$

Ex 4: find Z.T for $x(n) = a^n$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= 1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots$$

$$= \frac{1}{1 - a z^{-1}}$$

$$\boxed{\text{Roc: } a z^{-1} < 1}$$

$$X(z) = \frac{z}{z - a}$$

Ex 5: $x(n) = \sin \omega n$, find $X(z)$

Solution: for $x(n) = \sin \omega n = \frac{e^{j\omega n} - e^{-j\omega n}}{2j}$

$$X(z) = \sum_{n=0}^{\infty} \left[\frac{e^{j\omega n} - e^{-j\omega n}}{2j} \right] z^{-n}$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} [e^{j\omega n} - e^{-j\omega n}] z^{-n}$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} - \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n} \right]$$

$$= \frac{1}{2j} \left[(1 + e^{j\omega} z^{-1} + e^{j2\omega} z^{-2} + e^{j3\omega} z^{-3} + \dots) - (e^{-j\omega} z^{-1} + e^{-j2\omega} z^{-2} + e^{-j3\omega} z^{-3} + \dots) \right]$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{j\omega} z^{-1}} - \frac{1}{1 - e^{-j\omega} z^{-1}} \right]$$

$$= \frac{1}{2j} \left[\frac{-e^{-j\omega} z^{-1} + 1 + e^{j\omega} z^{-1}}{(1 + e^{j\omega} z^{-1})(1 - e^{-j\omega} z^{-1})} \right]$$

$$= \frac{1}{2j} \left[\frac{z^{-1} (e^{j\omega} - e^{-j\omega})}{1 - (e^{j\omega} + e^{-j\omega}) z^{-1} + z^{-2}} \right]$$

$$X(z) = \frac{z^{-1} \left(\frac{e^{j\omega} - e^{-j\omega}}{2j} \right)}{1 - z \left(\frac{e^{j\omega} + e^{-j\omega}}{2} \right) + z^{-2}}$$

$$= \frac{z^{-1} \sin \omega}{1 - z \cos \omega + z^{-2}}$$

$$X(z) = \frac{z \sin \omega}{z^2 - z \cos \omega + 1}$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Report
for $\cos \omega n$
(prove)

z - transform

$x(n)$	\longrightarrow $X(z)$
$\delta(n)$	1
$u(n)$	$\frac{z}{z-1}$
$e^{\pm an}$	$\frac{z}{z - e^{\pm a}}$
a^n	$\frac{z}{z-a}$
n	$\frac{z}{(z-1)^2}$
$\sin \omega n$	$\frac{z \sin \omega}{z^2 - 2z \cos \omega + 1}$
$\cos \omega n$	$\frac{z(z - \cos \omega)}{z^2 - 2z \cos \omega + 1}$

Properties of Z-transform:

$$\textcircled{1} \quad \mathcal{Z} \left[\underset{\text{const.}}{a} x(n) \right] = a X(z)$$

Ex: find Z.T for $[3u(n)]$

$$= \mathcal{Z.T} [3] = 3 \left(\frac{z}{z-1} \right)$$

$$\textcircled{2} \quad \mathcal{Z} [x_1(n) \pm x_2(n)] = X_1(z) \pm X_2(z)$$

$$\textcircled{3} \quad \mathcal{Z} [e^{\pm an} x(n)] = X(z) \Big|_{z = z e^{\mp a}}$$

Ex: find Z.T of $n e^{zn}$

$z = z e^{\mp a}$ \Rightarrow $z = z e^{\mp a}$

$$\mathcal{Z} [n e^{zn}] = \frac{z}{(z-1)^2} \Big|_{z = z e^{-2}}$$

$$= \frac{z e^{-2}}{(z e^{-2} - 1)^2}$$

$$\textcircled{4} \quad \mathcal{Z} [n x(n)] = -z \frac{dX(z)}{dz}$$

Ex: find Z.T n^2

$$\mathcal{Z} [n^2] = \mathcal{Z} [n \cdot \overset{x(n)}{n}] = -z \frac{d \left(\frac{z}{(z-1)^2} \right)}{dz}$$

$$\mathcal{Z} [n^2] = -z \frac{d \left(\frac{z}{(z-1)^2} \right)}{dz}$$

$$= -z \left[\frac{(z-1)^2(1) - z(z-1)}{(z-1)^4} \right]$$

$$= -z \left[\frac{(z-1) - z}{(z-1)^3} \right]$$

$$= -z \left[\frac{-z^{-1}}{(z-1)^3} \right] = \frac{z(z+1)}{(z-1)^3}$$

$$\textcircled{5} \quad z \left[a^n x(n) \right] = X(z) \Big|_{z = \frac{z}{a}}$$

$$\text{ex: } z \left[n a^n \right] \quad z = \frac{z}{a}$$

$$= \frac{z}{(z-1)^2} \Big|_{z = \frac{z}{a}} = \frac{z/a}{(z/a - 1)^2}$$

$$= \frac{a z}{(z a)^2}$$

Ex: $x(n) = a^n e^{2n}$, find $X(z)$

Sol 1: $e^{2n} \xrightarrow{z.T} \frac{z}{z-e^2}$

$$a^n a^{2n} \xrightarrow{z.T} \frac{z}{z-e^2} \Big|_{z = \frac{z}{a}}$$

$$= \frac{z/a}{z/a - e^2} = \frac{z}{z - a e^2}$$

Sol 2: $a^n \xrightarrow{z.T} \frac{z}{z-a}$

$$e^{2n} a^n \xrightarrow{z.T} \frac{z}{z-a} \Big|_{z = z e^{-2}}$$

$$= \frac{z e^{-2}}{z e^{-2} - a} = \frac{z}{z - a e^{+2}}$$

$\textcircled{6}$ Convolution in z -domain:

$$x_1(n) * x_2(n) \xrightarrow{z.T} X_1(z) \cdot X_2(z)$$

Ex: $x_1(n) = 3\delta(n) + z\delta(n+1)$
 $x_2(n) = 2\delta(n) - \delta(n-2)$

find $X(z) = z \left[x_1(n) * x_2(n) \right]$

Solution

$$X(z) = Z[x_1(n) * x_2(n)] = X_1(z) \cdot X_2(z)$$

$$\begin{cases} X_1(z) = 3 + 2z^{-1} \\ X_2(z) = 2 - z^{-2} \end{cases} \Rightarrow X(z) = X_1(z) \cdot X_2(z) = (3 + 2z^{-1})(2 - z^{-2})$$

$$\textcircled{7} \quad Z[x(n-m)] = z^{-m} X(z) \quad \text{Z.I.C}$$

Ex: $x(n) \{1, 0, 1, 0.5, 2\}$

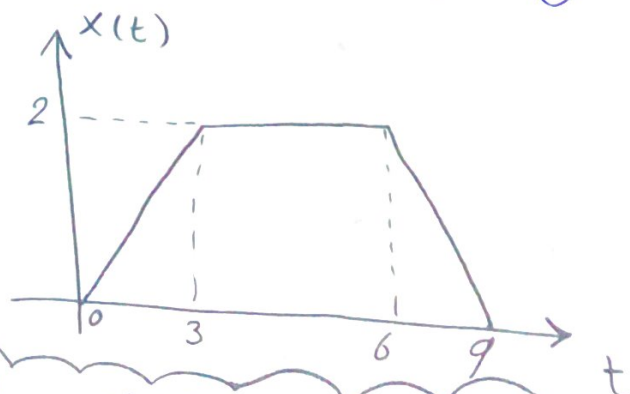
find $X(z)$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$

$$= 1 + z^{-2} + 0.5z^{-3} + 2z^{-4}$$

Report



find $X(z)$

$$T=1$$